4.3

The Power of Algebra Is a Curious Thing

Using Formulas to Determine Terms of a Sequence

LEARNING GOALS

In this lesson, you will:

- Write an explicit formula for arithmetic and geometric formulas.
- Write a recursive formula for arithmetic and geometric formulas.
- Use formulas to determine unknown terms of a sequence.

KEY TERMS

- index
- explicit formula
- recursive formula

Humans and tools go together like hand and glove. Some scientists claim that tools helped humans dominate other animals in ancient times. This makes sense—how else could humans have caught animals for food?

What about those ancient vegetarians? Yup, they used tools too—to till the soil and make it fertile for growing crops. Humans use tools for mathematics as well. Algebraic thinking has been around for centuries—in fact, it has been around for such a long period of time that two different people are associated with being the "father of algebra." No matter, algebra led to other tools like the abacus, the modern graphing calculator, and even the computer.

In much the same way, formulas were some of the first "tools" used to help humans calculate more quickly. And you probably guessed it: formulas can help you determine any unknown term in a sequence!

PROBLEM 1 Can I Get a Formula?

While a common ratio or a common difference can help you determine the next term in a sequence, how can they help you determine the thousandth term of a sequence? The ten-thousandth term of a sequence? Consider the sequence represented in the given problem scenario.

1. Rico owns a sporting goods store. He has agreed to donate $125 to the Centipede Valley High School baseball team for their equipment fund. In addition, he will donate $18 for every home run the Centipedes hit during the season. The sequence shown represents the possible dollar amounts that Rico could donate for the season.

125, 143, 161, 179, . . .

a. Identify the sequence type. Describe how you know.

Notice that the 1st term in this sequence is the amount Rico donates if the team hits 0 home runs.

b. Determine the common ratio or common difference for the given sequence.

c. Complete the table of values. Use the number of home runs the Centipedes could hit to identify the term number, and the total dollar amount Rico could donate to the baseball team.

Number of Home Runs	Term Number (*n*)	Donation Amount (dollars)
0	1	
1		
2		
3		
4		
5		
6		
7		
8		
9		

d. Explain how you can calculate the tenth term based on the ninth term.

e. Determine the 20th term. Explain your calculation.

f. Is there a way to calculate the 20th term without first calculating the 19th term? If so, describe the strategy.

4

g. Describe a strategy to calculate the 93rd term.

You can determine the 93rd term of the sequence in Question 1 by calculating each term before it, and then adding 18 to the 92nd term, but this will probably take a while! A more efficient way to calculate any term of a sequence is to use a formula.

Analyze the table. The examples shown are from the sequence showing Rico's contribution to the Centipedes baseball team in terms of home runs hit.

General Rule	Example
A lowercase letter is used to name a sequence.	a
The first term, or initial term, is referred to as a_1.	$a_1 = 125$
The remaining terms are named according to the term number.	$a_2 = 143$, $a_3 = 161, \ldots$
A general term of the sequence is referred to as a_n, also known as the *n*th term, where n represents the *index*.	a_n
The term previous to a_n is referred to as a_{n-1}.	a_{n-1}
The common difference is represented as d.	$d = 18$

The **index** is the position of the term (its term number) in a sequence.

2. What is a_3 in the sequence representing Rico's possible donation amount?

From these rules, you can develop a formula so that you do not need to determine the value of the previous term to determine subsequent terms.

An **explicit formula** of a sequence is a formula for calculating the value of each term of a sequence using the term's position in the sequence. The explicit formula for determining the *n*th term of an arithmetic sequence is:

$$\overset{n\text{th term}}{a_n} = \underset{\text{1st term}}{a_1} + \overset{\text{common difference}}{d}\underbrace{(n-1)}_{\text{previous term number}}$$

Consider the explicit formula to determine the 93rd term in this problem situation.

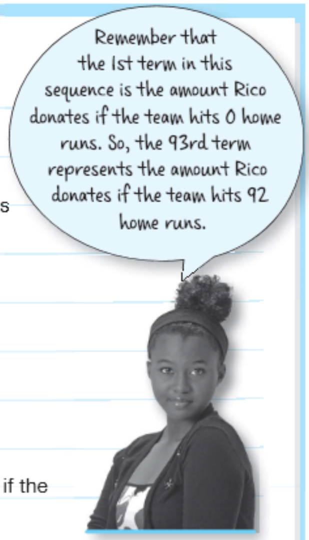
$$a_n = a_1 + d(n - 1)$$
$$a_{93} = 125 + 18(93 - 1)$$

where a_{93} represents the 93rd term, a_1 represents the first term (which is 125), the common difference d is 18, and the previous term from 93 is (93 − 1).

$$a_{93} = 125 + 18(92)$$
$$a_{93} = 125 + 1656$$
$$a_{93} = 1781$$

The 93rd term of the sequence is 1781.

This means Rico will contribute a total of $1781 if the Centipedes hit 92 home runs.



3. Use the explicit formula to determine the amount of money Rico will contribute if the Centipedes hit:

a. 35 home runs.

b. 48 home runs.

c. 86 home runs.

d. 214 home runs.

4. Rico decides to increase his initial contribution and amount donated per home run hit. He decides to contribute $500 and will donate $75.00 for every home run the Centipedes hit. Determine Rico's contribution if the Centipedes hit:

   a. 11 home runs.

   b. 26 home runs.

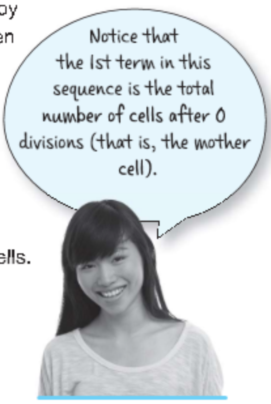
   c. 39 home runs.

   d. 50 home runs.

   e. Write the first 10 terms of the sequence representing the new contribution Rico will donate to the Centipedes.

PROBLEM 2 They're Just Out of Control—But That's A Good Thing!

When it comes to bugs, bats, spiders, and—ugh, any other creepy crawlers—finding one in your house is finding *one* too many! Then again, when it comes to cells, the more the better! Animals, plants, fungi, slime, molds, and other living creatures consist of eukaryotic cells. During growth, generally there is a cell called a "mother cell" that divides itself into two "daughter cells." Each of those daughter cells then divides into two more daughter cells, and so on.



1. The sequence shown represents the growth of eukaryotic cells.

   1, 2, 4, 8, 16, . . .

   a. Describe why this sequence is geometric.

b. Determine the common ratio for the given sequence.

c. Complete the table of values. Use the number of cell divisions to identify the term number, and the total number of cells after each division.

Number of Cell Divisions	Term Number (*n*)	Total Number of Cells
0	1	
1		
2		
3		
4		
5		
6		
7		
8		
9		

d. Explain how you can calculate the tenth term based on the ninth term.

e. Determine the 20th term. Explain your calculation.

f. Is there a way to calculate the 20th term without first calculating the 19th term? If so, describe the strategy.

As you discovered in Problem 1, *Can I Get a Formula?* a more efficient way to calculate any term of an arithmetic sequence is to use an explicit formula. You can also use an explicit formula for geometric sequences.

Analyze the table shown. The examples are from the sequence showing eukaryotic cell growth.

General Rule	Example
A lowercase letter is used to name a sequence.	g
The first term, or initial term, is referred to as g_1.	$g_1 = 1$
The remaining terms are named according to the term number.	$g_2 = 2$, $g_3 = 4, \ldots$
A general term of the sequence is referred to as g_n, also known as the *n*th term.	g_n
The term previous to g_n is referred to as g_{n-1}.	g_{n-1}
The common ratio is represented as r.	$r = 2$

2. What is g_3 in the sequence representing eukaryotic cell growth?

The explicit formula for determining the *n*th term of a geometric sequence is:

$$\overset{\text{nth term}}{g_n} = \underset{\text{1st term}}{g_1} \cdot \underset{\text{common ratio}}{r}^{\overbrace{n-1}^{\text{previous term number}}}$$

Consider the explicit formula to determine the 20th term in this problem situation.

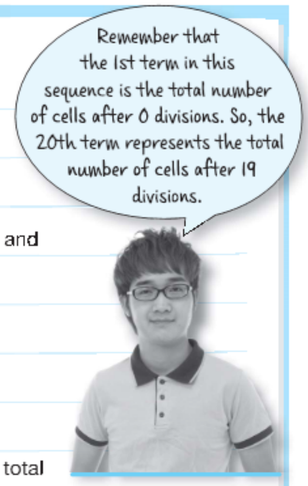
$$g_n = g_1 \cdot r^{n-1}$$
$$g_{20} = 1 \cdot 2^{20-1}$$

where g_{20} represents the 20th term, g_1 represents the first term (which is 1), the common ratio r is 2, and $^{20-1}$ represents the previous term number.

$$g_{20} = 1 \cdot 2^{19}$$
$$g_{20} = 1 \cdot 524{,}288$$
$$g_{20} = 524{,}288$$

The 20th term of the sequence is 524,288.

This means that after 19 cell divisions, there are a total of 524,288 cells.



3. Use the explicit formula to determine the total number of cells after:

a. 11 divisions.

b. 14 divisions.

c. 18 divisions.

d. 22 divisions.

4. Suppose that a scientist has 5 eukaryotic cells in a petri dish. She wonders how the growth pattern would change if each mother cell divided into 3 daughter cells. For this situation, determine the total number of cells in the petri dish after:

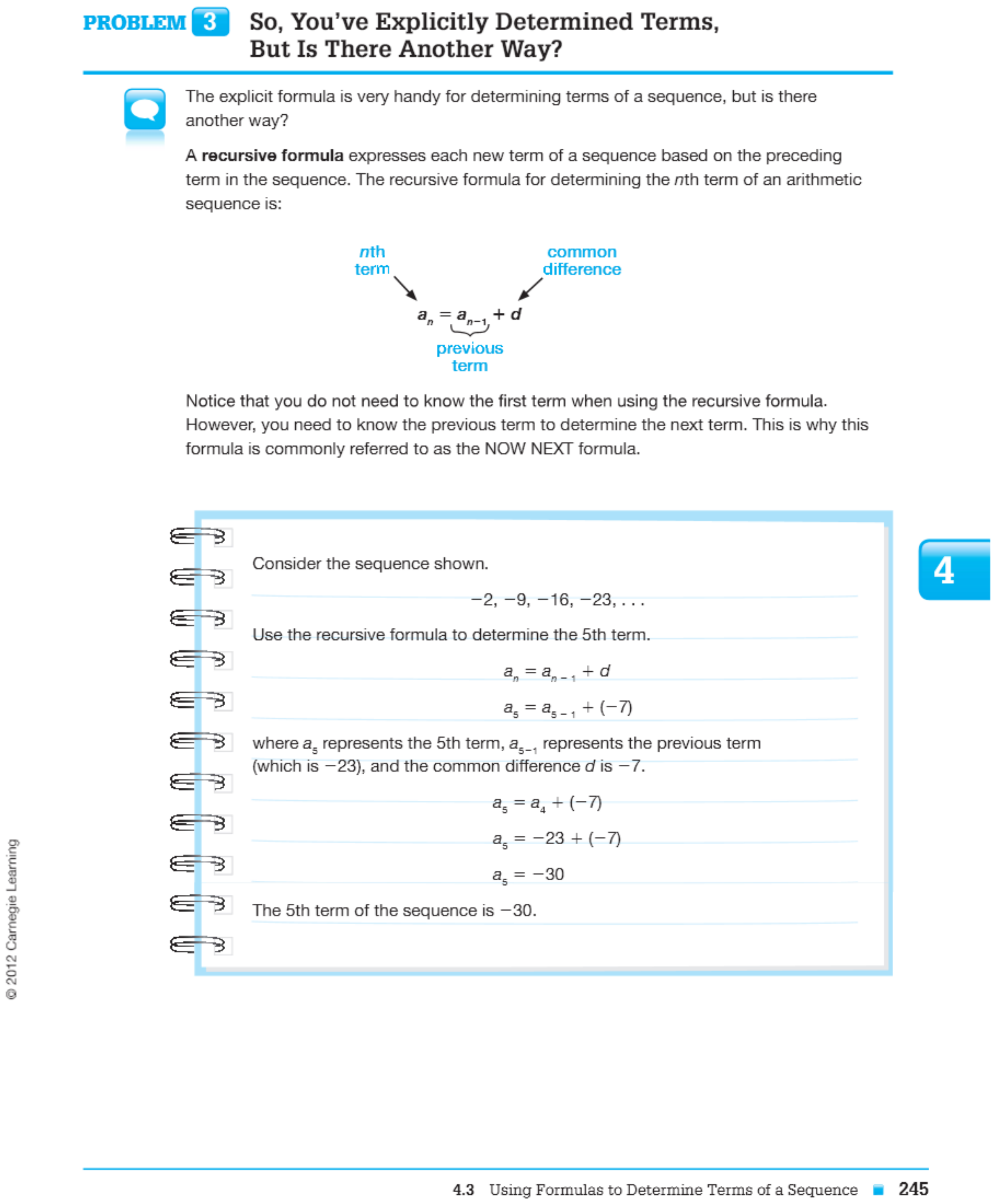
a. 4 divisions.

b. 7 divisions.

c. 13 divisions.

d. 16 divisions.

e. Write the first 10 terms of the sequence for the scientist's hypothesis.



PROBLEM 3 So, You've Explicitly Determined Terms, But Is There Another Way?

The explicit formula is very handy for determining terms of a sequence, but is there another way?

A **recursive formula** expresses each new term of a sequence based on the preceding term in the sequence. The recursive formula for determining the *n*th term of an arithmetic sequence is:

*n*th term

common difference

$$a_n = \underbrace{a_{n-1}}_{\text{previous term}} + d$$

Notice that you do not need to know the first term when using the recursive formula. However, you need to know the previous term to determine the next term. This is why this formula is commonly referred to as the NOW NEXT formula.

Consider the sequence shown.

$$-2, -9, -16, -23, \dots$$

Use the recursive formula to determine the 5th term.

$$a_n = a_{n-1} + d$$

$$a_5 = a_{5-1} + (-7)$$

where a_5 represents the 5th term, a_{5-1} represents the previous term (which is −23), and the common difference *d* is −7.

$$a_5 = a_4 + (-7)$$

$$a_5 = -23 + (-7)$$

$$a_5 = -30$$

The 5th term of the sequence is −30.

4

The recursive formula for determining the *n*th term of a geometric sequence is:

*n*th term common ratio

$$\overset{n\text{th term}}{g_n} = \underbrace{g_{n-1}}_{\text{previous term}} \cdot \overset{\text{common ratio}}{r}$$

Consider the sequence shown.

4, 12, 36, 108, . . .

Use the recursive formula to determine the 5th term.

$$g_n = g_{n-1} \cdot r$$

$$g_5 = g_{5-1} \cdot (3)$$

where g_5 represents the 5th term, g_{5-1} represents the previous term (which is 108), and the common ratio *r* is 3.

$$g_5 = g_4 \cdot (3)$$

$$g_5 = 108 \cdot (3)$$

$$g_5 = 324$$

The 5th term of the sequence is 324.

1. Determine whether each sequence is arithmetic or geometric. Then use the recursive formula to determine the unknown term in each sequence.

a. $\frac{5}{3}$, 5, 15, 45, ________ , . . .

b. −45, −61, −77, −93, ________ , . . .

c. −3, 1, ________ 9, 13, . . .

d. −111, 222, ________, 888, −1776, . . .

e. −30, −15, ________, −3.75, −1.875, ________, . . .

f. 3278, 2678, 2078, ________, ________, ________, . . .

2. Consider the sequence in Question 1 part (f).

a. Use the recursive formula to determine the 9th term.

b. Use the explicit formula to determine the 9th term.

c. Which formula do you prefer? Why?

d. Which formula would you use if you wanted to determine the 61st term of the sequence? Explain your reasoning.

Let's explore how to use a recursive formula on a graphing calculator to determine the 20th term in sequence *t*.

Consider the sequence *t*: 3, 10, 17, 24, 31, . . .

You can use a graphing calculator to generate terms in a sequence using a recursive formula.

Step 1: Enter the first value of the sequence. Then press **ENTER** to register the first term. The calculator can now recall that first term.

Step 2: From that term, add the common difference. Press **ENTER**. The next term should be calculated. The calculator can now recall the formula as well.

Step 3: Press **ENTER** and the next term should be calculated.

Step 4: Continue pressing **ENTER** until you determine the *n*th term of the sequence you want to determine.

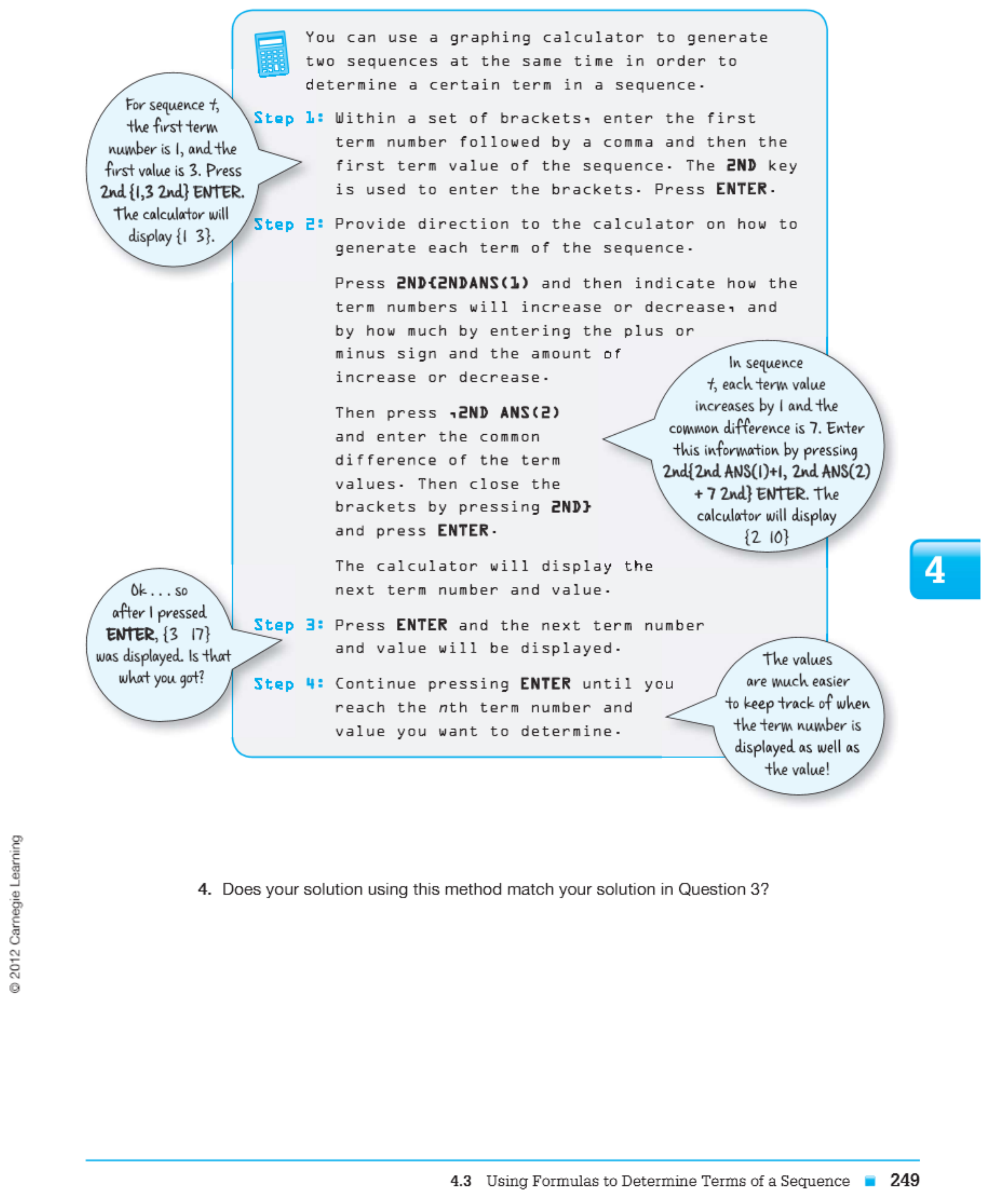
Press 3 then ENTER since 3 is the first term.

The common difference is 7.

Keep track of how many times you press ENTER so you know when you have the 20th term!

3. Determine the 20th term of sequence *t*. How did you determine this term?

The given calculator instructions can help you identify the term of a sequence. However, you might have found it challenging to keep track of the term numbers when calculating the 20th term. Another way to determine the 20th term in sequence *t* is to use a graphing calculator to generate 2 sequences at the same time. The first sequence will keep track of the term number, and the second sequence will generate the term value.



You can use a graphing calculator to generate two sequences at the same time in order to determine a certain term in a sequence.

For sequence *t*, the first term number is 1, and the first value is 3. Press **2nd {1,3 2nd} ENTER.** The calculator will display {1 3}.

Step 1: Within a set of brackets, enter the first term number followed by a comma and then the first term value of the sequence. The **2ND** key is used to enter the brackets. Press **ENTER**.

Step 2: Provide direction to the calculator on how to generate each term of the sequence.

Press **2ND{2NDANS(1)** and then indicate how the term numbers will increase or decrease, and by how much by entering the plus or minus sign and the amount of increase or decrease.

Then press **,2ND ANS(2)** and enter the common difference of the term values. Then close the brackets by pressing **2ND}** and press **ENTER**.

In sequence *t*, each term value increases by 1 and the common difference is 7. Enter this information by pressing **2nd{2nd ANS(1)+1, 2nd ANS(2) + 7 2nd} ENTER.** The calculator will display {2 10}

The calculator will display the next term number and value.

Step 3: Press **ENTER** and the next term number and value will be displayed.

Ok . . . so after I pressed **ENTER**, {3 17} was displayed. Is that what you got?

Step 4: Continue pressing **ENTER** until you reach the *n*th term number and value you want to determine.

The values are much easier to keep track of when the term number is displayed as well as the value!

4. Does your solution using this method match your solution in Question 3?

5. Use a graphing calculator to determine each solution.
   - **a.** Identify the seventh term of this arithmetic sequence: 6, 14, 22, . . .
   - **b.** List the first 10 terms of this arithmetic sequence: 54, 47, 40, . . .
   - **c.** List the first 10 terms of the arithmetic sequence generated by this recursive formula:

     $t_1 = 8$

     $t_n = t_{n-1} + 19$
   - **d.** Identify the 30th term of this arithmetic sequence: 45, 51, 57, . . .?

Talk the Talk

1. Explain the advantages and disadvantages of using the explicit formula.

4

2. Explain the advantages and disadvantages of using the recursive formula.

Be prepared to share your solutions and methods.